Trakia Journal of Sciences, Vol. 9, Suppl. 2, pp 302-309, 2011

Available online at:
http://www.uni-sz.bg

ISSN 1313-7069 (print)
ISSN 1313-3551 (online)

OPTION PRICING IN THE JUMP DIFFUSION FRAMEWORK: A RADIAL BASIS FUNCTIONS APPROACH

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ABSTRACT

In this paper we consider a partial integro-differential equation (PIDE) problem with a free boundary, arising in American option model when the stock price follows a diffusion process with jump components. We use a front-fixing transformation of the underlying asset variable to fix the free boundary conditions. We present the radial basis functions (RBFs) method and we point its advantages for pricing of options by comparing it with the finite difference schemes for parabolic equations as the Black-Scholes one. We obtained accurate results of the presented method are in agreement with those obtained by other quantitative methods in literature for pricing American options.

Mathematics Subject Classification 2010: Primary: 65M06, 35R09, 35K.55; Secondary: 91B25, 91G60

Key words: American Options, Jump Diffusion Process, Finite Difference Schemes, Poisson Distribution, Log-normal distribution, Radial Basis Functions

INTRODUCTION

There is a great bulk of literature and heuristic evidences showing that the normal distribution assumption for returns is a poor fit to data, in part because the observed area in the tails of the distribution is much greater than the normal distribution permits [13], [14]. If the probability distribution of the stock price at any given future time is not lognormal then we underprice or overprice call and put options, depending on the distributions's tails [10]. One heuristic evidence is the behavior of underlying assets in real options modeling such as energy prices, oil and natural gas prices. These quantities can exhibit peaks and spikes, for example, in one case the unit price of natural gas jumped from 30 euros to more than 1500 euros in one day during a period of short supply [3]. One possible remedy is to assume an alternative distribution for the returns of the asset price that, like the normal distribution, is infinitely divisible but that has more area in the tails. For example, the inverse Gaussian (NIG) distribution is a good fit to some stock and interest rate data [12], [14]. Another idea to overcome the shortcomings of the inaccurate pricing of the contingent claims is using more appropriate models such as stochastic volatility models (Fleston model) or models where the stock price may experience occasional jumps rather than the continuous changes. As a consequence, in contrast to the original Black-Scholes framework where for most of the option valuation problems closed-form formulas exist or standard numerical methods could be applied, in nonstandard financial models more complicated and precise techniques are required. We have chosen to extend the Black-Scholes model, by exploring the jump diffusion (Poisson) model, see Merton [15] and radial basis function for approximation of the option prices.

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In Section 2 we discuss the original Black-Scholes model and expose the mathematical model for pricing American option under jump-diffusion process. We postulate the option valuation problem as a partial integral-differential equation that is an extension of the famous Black-Scholes equation with an integral part that reflects the jumps of the asset price. We apply a front-fixing transformation of the PDE part and give a numerical recipe for the semi bounded integral part.

In Section 3 we point out the main features of the finite difference schemes as this is one of the most explored numerical methods in computational finance and origin of numerical techniques such as operator splitting and predictor-corrector methods. In contrast to the Monte Carlo method and like finite difference schemes (FDS), the RBFs method treats the option valuation problem without simulating the random movement of the asset price. We make a parallel between the FDS and the structure and application of the RBFs method. We discuss the numerical accuracy of the RBFs method.

In Section 4 we expose the main idea how the radial basis functions method should be applied in order to be resolved the option valuation problem in case of pricing American options in the jump-diffusion framework. A detailed explanation how the radial basis functions method is applied and how the option valuation problem is reduced to a solving a standard non-singular square linear system is described by Milev in [21].

In the conclusion, we give some final remarks for the advantages of the presented radial basis function method and its possible applications.

MATHEMATICAL MODEL: THE MODIFIED BLACK-SCHOLES PDE

Usually in financial literature, as a mathematical model for the movement of the asset price under the risk-neutral measure is considered a standard *geometric Brownian motion* diffusion process with constant coefficients r and σ:

$$dS/S - r\,dt + \sigma\, dW_t \qquad (1)$$

The contract to be priced is a discretely monitored double barrier knock-out call option. If τ is the time to expiry T of the contract, i.e. $\tau = T - t$, $0 \le t \le T$, the price $V(S,t)$ of the option satisfies the Black-Scholes partial differential equation

$$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \qquad (2)$$

In this article we explore a more general model for the random movement of the asset price. In order to make our analysis concrete, we concentrate our attention on a classical problem in financial literature, i.e valuation of american put options under jump diffusion. The modified stochastic differential equation that models jumps is:

$$dS/S = r\,dt + \sigma\, dW_t + (\eta - 1)dq \qquad (3)$$

where S is the underlying stock price, r - interest rate, σ - volatility, dW_t increments of Gauss-Wiener process, dq - Poisson process with arrival rate (intensity) λ defined as:

$$dq = \begin{cases} 0 & \text{with probability } 1-\lambda dt \\ 1 & \text{with probability } \lambda\, dt \end{cases}$$

and $\eta - 1$ impulse function producing a jump from S to $S\eta$ where $k = E(\eta - 1)$ is the expected relative jump size.

In contrast to equation (1) in the new equation (3) the path followed by S is continuous most of the time while *finite negative or positive jumps* will appear at discrete points in time. As a result the Black-Scholes equations (2) is modified as a partial integral differential equation by adding an integral part that reflects the jump diffusion structure of the process [5]:

$$-\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k)S\frac{\partial V}{\partial S} - (r+\lambda)V + \lambda\int_0^\infty V(S\eta)\rho(\eta)\,d\eta \qquad (4)$$

endowed with initial and boundary conditions:

$$V(S,T) = max(K - S, 0) \qquad (5)$$

$$V(S,t) \to 0 \text{ as } S \to \infty \text{ and } V(0,t) = K \qquad (6)$$

and with the additional condition for S

$$\lim_{S \to b(t)} V(S,t) = K - b(t) \tag{7}$$

where η shows the jump amplitude and the function ρ satisfies

$$\rho(\eta) \geq 0, \qquad \int_0^\infty \rho(\eta) d\eta = 1, \quad \rho(\eta) = \frac{1}{\sqrt{2\pi}\sigma_J \eta} e^{\frac{-(\ln\eta - \mu)^2}{2\sigma_J^2}} \tag{8}$$

where μ is the mean of lognormal jump process and $b(t)$ is the optimal exercise boundary which should be approximated. It is known that only the region $S > b(t)$ has financial meaning. The financial parameters $r, \sigma, \lambda, k, K, \sigma_J, \eta$ are constant numbers in this mathematical model. The big letter K is the strike price of the option contract. Obviously, if the parameter $\lambda = 0$, i.e. the parameter that characterizes the jumps of the asset price is zero, then the new equation (3) is the standard Black-Scholes equation (1), having in mind that $\tau = T - t$. After rescaling the variables:

$$S = K\hat{S}, \quad P_A(S,\tau) = \hat{P}_A(\hat{S},\tau), \quad b(t) = K\hat{b}(t) \tag{9}$$

and using front fixing method by applying the transformation $y = \ln \frac{\hat{S}}{\hat{b}(t)}$ in equation (4) so that the free boundary $\hat{b}(t)$ associated with the optimal exercise prices is converted into a fixed boundary [9]. Thus when $\tau = T - t$, and defining $\hat{P}_A(\hat{S},\tau) = V(x,\tau)$, the problem of pricing an American put option becomes

$$\frac{\partial V}{\partial \tau} - \frac{1}{b(\tau)}\frac{db(\tau)}{d\tau}\frac{\partial V}{\partial y} = \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{\partial y}\right) + (r - \lambda k)\frac{\partial V}{\partial y} - (r+\lambda)V + \frac{\lambda}{b(\tau)e^y}\int_0^\infty V(u)\rho\left(\frac{u}{b(\tau)e^y}\right)du \tag{10}$$

finally by using (8), the equation (10) reduces to the following form:

$$\frac{\partial V}{\partial \tau} - \frac{1}{b(\tau)}\frac{db(\tau)}{d\tau}\frac{\partial V}{\partial y} = \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{\partial y}\right) + (r - \lambda k)\frac{\partial V}{\partial y} - (r+\lambda)V + \lambda\int_0^\infty \frac{V(u)}{u}\exp\left(\frac{-\ln(u-\mu)^2}{2Kb(\tau)\exp(x)\sigma_J^2}\right)du \tag{11}$$

And

$$\left| \begin{array}{l} V(y,0) = \max(1 - b(0)e^y, 0) = 0, \ for\ 0 < y < \infty, \ as\ b(0) = 1 \\ V(0,\tau) = 1 - b(\tau) \\ \dfrac{\partial V(0,\tau)}{\partial y} = -b(\tau) \\ \lim\limits_{y \to \infty} V(y,\tau) = 0 \end{array} \right. \tag{12}$$

Frequently $b(r)$ is isolated from conditions (12) and we have

$$\begin{cases} V(y,0)=0, & for\ 0<y<\infty \\ V(0,\tau)-\dfrac{\partial V(0,\tau)}{\partial y}=1 \\ \lim\limits_{y\to\infty} V(y,\tau)=0 \end{cases} \tag{13}$$

The key advantage of the obtained model is that equation (11) permits us to apply finite difference schemes and the *radial basis functions method* without using the *linear complementarity problem* formulation for valuation of American options.

We note also that equation (11) is non-linear having the non-linear term $\dfrac{1}{b(\tau)}\dfrac{db(\tau)}{d\tau}\dfrac{\partial V}{\partial y}$.

As the equation (11) is partial integral-differential equation (PIDE), i.e. it consists of a partial differential equation (PDE) part and an integral part, the solution of the this problem for valuation of American put options will be divided into two steps:

1. Approximation of the integral part of equation (11);
2. Implementation of the radial basis function for the PDE part;

Duffy in [3] proposes a modified form of equation (4) in which the integrand is defined on a bounded interval:

$$\frac{\partial u}{\partial t}=Lu+\lambda\int_A^B u(x+y,t)\Gamma_\delta(y)dy$$

Another similar question is the that the PDE part of equation (11) is also defined on a semi-infinite interval $[0,\infty)$, but this problem is easily resolved having in mind the variable 5 is the underlying asset value. The S -domain is truncated at the value S_{max} = 3K, sufficiently large such that computed values are not appreciably affected by the upper boundary. As in financial practice is well-known that $S \geq 3K$, (K is the strike price), has no financial interest, then S_{max} = 3K is enough to be chosen. In finite difference practice usually, the $S_{max} = 2K$. For more precise valu.es of the values of S_{max} are explored in [8].

RADIAL BASIS FUNCTIONS APPROACH

1 Comparison with the Finite Difference Method

As usual, in the finite difference approximation the S -domain is truncated at the value S_{max}, sufficiently large such that computed values are not appreciably affected by the upper boundary. The computational domain $[0, S_{max}] \times [0, T]$ is discretized by a uniform mesh with steps *AS*, *At.* Therefore we obtain the nodes S_j and t_n, where

$(S_j = j\Delta S, t_n = n\Delta t)$, j = 0,K ,n

n = 0,K , N_1 so that $S_{max} = N\Delta S$, $T = N_1\Delta t$, N_1 and N are integer numbers.

1. The choice of a specific numerical scheme is based on its property of convergence. The requirement rests on the Lax equivalence theorem.
2. The parabolic nature of the Black-Scholes equation ensures that being the initial condition $V(S,0) = (S - K)^+ 1_{[L,U]}(S)$ square-integrable the solution is smooth in the sense that $V(\bullet,t) \in C^\infty(R^+)$, $\forall t \in [t_{i-1}, t_i^-]$, $i =$ K, F.

Thus rough initial data give rise to smooth solutions in infinitesimal time.

As there are not still a closed-form solution of equation (11), we will point most of the frequently used finite difference methods in Numerical analysis that are listed in the book of Duffy [3].

1. Explicit and implicit numerical schemes - the so called Θ -method;
2. Implicit-explicit Runge-Kutta methods;
3. Operator splitting methods;
4. Predictor-Corrector methods;

We will not enter in details for advantages and disadvantages of the above listed numerical methods, but we would like to point out some practical problems that we would like to avoid such as *conditional stability [25], non-smooth solutions and undesired spurious oscillations [18], lower-order of accuracy [19], unreasonable computational time [17], artificial numerical diffusion [20], errors induced by splitting [3] and impossibility of* *multi-dimensional implementation [11], [12], [13], [22].*

2 Advantages of Radial Basis Functions Approach

We listed in the introduction the following three main advantages of the radial basis functions (RBFs) method that could be observed from the structure and way of application of this approach:

1. The RBFs is an extremely flexible interpolation method because does not depend on the locations of the approximation nodes and applies immediately for scattered data. Thus, unlike finite difference schemes, application and proving convergence do not differ in case of uniform and non-uniform grids. Often the RBFs method is referred to meshless methods [23].
2. The RBFs generalization in any number of dimensions is immediate and analogous to the one dimensional case. Thus pricing multi-asset option problems is significantly simplified and does not seem too arduous task as it is generally concerned in computational finance [11], [22].
3. The RBFs method depends on a *shape parameter* [4] which by adjusting it, we can have higher accuracy.

We have explored in details the application of RBFs method for different values of the shape parameter in [21].

In the previous subsection 3.1, we have described the main features of the finite difference schemes (FDS). Let us expose the main idea of radial basis functions method [1] in order to make a parallel with the finite difference schemes. Let us at each data location x_j, $i = 1, K, n$ place a translate of the function $\phi(x) = |x|^3$, i.e. x. the function $\phi(x - x_i) = \|x - x_i\|^3$. We then try, if it is possible, to form a linear combination of all these functions.

$$V(x,\tau) = \sum_{j=0}^{\infty} \lambda_j(\tau)\phi(u - u_j) \qquad (14)$$

In d dimensions, we use the rotated version of the same radial function and we could write them as $\phi(\|x - x_i\|)$, where $\|\cdot\|$ denotes the standard Euclidean norm. In particular, we note that the algebraic complexity of the interpolation problem has not increased with the number of dimensions, we will always end up with a square symmetric system of the same size as the number of data points. Cubic splines have thus been generalized to apply also to scattered data in any number of dimensions. For different types of radial basis functions with their corresponding names and respectively, the properties of each type could be found in [1].

DISCRETIZATION BY RADIAL BASIS FUNCTIONS

In this section we will only describe briefly the procedure to develope an *efficient and accurate numerical scheme* for pricing American options:

1. We will interpolate the integral term by the Laguerre polynomials.
2. We use the radial basis functions (RBFs) as a flexible interpolation method and we achieve a *system of nonlinear equations.*
3. To overcome the nonlinearity of the system, we apply Cranck-Nicolson and Predictor-Corrector method as well as to reach stable and correct results.

We have described in details the implementation of the presented algorithm of radial basis functions in [21] and we have explored the convergent error for different values of the underlying asset value. We have compared our results with those obtained by other numerical schemes to show the validity of RBFs method.

NUMERICAL RESULTS

The presented procedure of radial basis fuctions has a simple computer implementation [21] similarly the approach of the finite difference schemes but in contrast to them it is much more faster [23] where the RBFs method turns out to be 20-40 times faster in one and two space dimensions and has approximately the same memory requirements compared with an adaptive finite difference method. This is confirmed by the numerical results in Table 1 where we have done the same experiment as in [6]. We note also the small number of time steps and size S -grid of the RBFs method. We observe a very fast convergence of the RBFs method and we obtain a third-order accuracy with only 40 timestpes and 15 space nodes. **(Table 1)**

Table 1. Input data used to value American options under the lognormal jump diffusion process

Underlying asset price and jump diffusion parameter values						
Asset Price Parameters		Jump Parameters			Fix Contract Parameters	
Volatility σ	Interest Rate r	Jump Volatility σ_J	Mean μ	Jumps Intensity λ	Maturity T	Strike Price K
0.15	0.05	0.45	-0.90	0.10	0.25	100

In this section we present radial basis functions method results exploring the numerical examples for pricing American put options in [6]. We show the error convergence in Figure 1 according to the variation of shape parameter, stock mesh size when number of time steps is fixed considering the parameters in Table 1 for all experimets. We obtain the value of the American put option prices for three different stock prices at S_0 =90, 100, 110 and compute the error value for each of them.

We use for comparison the reference values in [6] that are respectively 10.003822 at $S_0=90$, 3.241251 at S_0=100 and 1.419803 at S_0=110.

We see from Table 2 that our results are in agreement till 3-rd to 4-th decimal point.

We point in Figure 1 the optimal values of the shape parameter c when the solution error reaches to its minimal value. **(Figure 1, Table 2)**

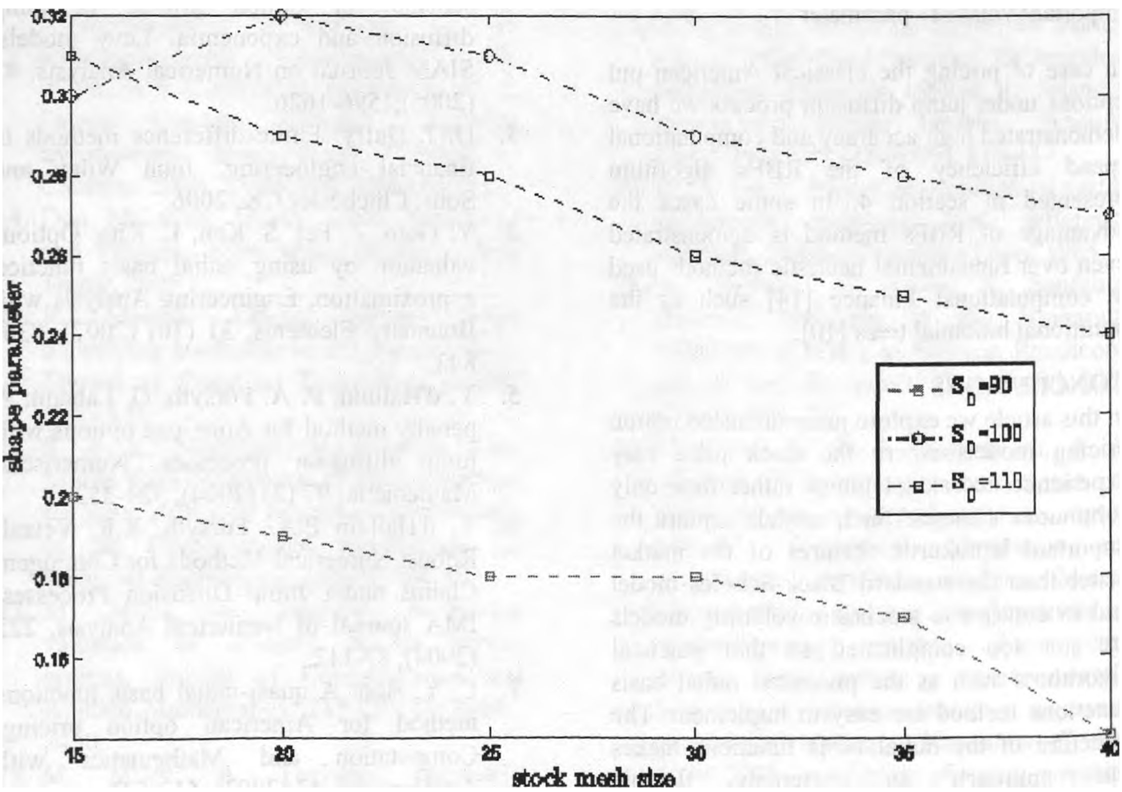

Figure 1. The variation of shape parameter with respect to the stock mesh size by fixing the time mesh (N_1=40)

Table 2. American put option values using a Predictor-Corrector scheme and the Crank-Nicolson method

Number of time steps	Size of the space S-grid	S_0=90		S_0=100		S_0=110	
		Value	space S-grid	Value	Error	Value	Error
40	15	10.007723	15	1.424355	4×10^{-3}	3.23832	2×10^{-3}
50	20	10.003251	20	1.410948	8×10^{-3}	3.239984	1×10^{-3}
60	25	10.003547	25	1.4198438	4×10^{-4}	3.2406385	6×10^{-4}
70	30	10.003968	30	1.4200659	2×10^{-4}	3.2402608	9×10^{-4}
80	35	10.0038211	35	1.418803	3×10^{-4}	3.2413965	1×10^{-4}

Most articles in literature describe and confirm the validity of the RBFs method for pricing European, barrier and Asian options of single asset in the original Black-Scholes framework or under the so called pure log-normal process [4], [7], [22], [23]. Using the RBFs approach and a computer simulation of investment portfolio both within Matlab environment as in [26] we could estimate and predict the important volatility parameter σ.

In case of pricing the classical American put options under jump diffusion process we have demonstrated high accuracy and computational speed efficiency of the RBFs algorithm presented in section 4. In some cases the advantage of RBFs method is demonstrated even over fundamental heuristic methods used in computational Finance [14] such as the traditional binomial trees [10].

CONCLUSIONS

In this article we explore jump diffusion option pricing models where the stock price may experience occasional jumps rather than only continuous changes. Such models capture the important leptokurtic features of the market better than the standard Black-Scholes model and in contrast to stochastic volatility models are not too complicated so that practical algorithms such as the presented radial basis functions method are easy to implement. The structure of the radial basis functions makes this approach an extremely flexible interpolation method avoiding most of the frequently met problems in computational finance such as the unstable or slow convergent numerical solutions, multi-asset valuation, complexity of computer implementation, unreasonable time and memory requirements.

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